INVESTMENT AND DYNAMIC DEA

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Abstract
A dynamic version of Data Envelopment Analysis (DEA) is developed in the present paper. Our model introduces investment in traditional DEA and imposes intertemporal cost minimization. Adding an intertemporal adjustment constraint into the cost minimization problem, we derive the relation between the DEA variables of the cost function and those of the primary production frontiers' coefficients. The augmented DEA model can be solved using standard linear programming. This dynamic framework enables computing the production frontiers, measuring the productive efficiencies and evaluating the potential economies all in the presence of adjustment costs.

JEL Classification: D24, L23

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1. Introduction

The data envelopment analysis (DEA), pioneered by Charnes, Cooper, and Rhodes (1978), is an alternative to the traditional parametric approach of quantitative analysis. It offers a robust tool in studying of production frontiers and evaluating the performance of decision-making units (DMU). DEA is an optimization method based on mathematical programming generalizing the Farrell (1957) single-input/single-output technical efficiency measure to the multiple-input/multiple-output case by constructing a relative efficiency score as the ratio of a single virtual output to a single virtual input.

DEA has rapidly extended to returns to scale, dummy or categorical variables, discretionary and non-discretionary variables, incorporating value judgments, longitudinal analysis, weight restrictions, stochastic DEA, non-parametric Malmquist indexes, technical change in DEA and many other features. But until now, most existing DEA models rarely take into account intertemporal adjustment costs. Consequently, the estimated frontiers may be biased, even strongly biased in presence of high adjustment costs. To correctly calculate the production frontier, the adjustment restrictions and costs from period to period must be taken into account. Nemoto and Goto (2003) proposed a DEA model with adjustment costs incorporated. The authors implicitly impose the following assumption on the adjustment: when investments occur, the fictive "*best*" DMU's adjustment cost is often strictly higher than that of the considered DMU. Because of this, the efficiency of the considered DMU could be overestimated. When disinvestments occur, the fictive dominant DMU has an adjustment cost lower than that of the considered DMU, which is insufficient to be *best* DMU and the efficiency score so obtained could be anything. Briefly, biased results could rise from the authors' irrelevant assumption.

The present paper focuses on introducing a new version of dynamic DEA model with weaker and more realistic assumptions.

First, we define a *temporary* production function. By temporary, we mean the production frontier existing at a given time. The term frontier refers to an optimal capacity of transformation of the inputs into outputs. Several definitions are possible. We can define the frontier by the maximal quantity that a production unit can produce from the given quantities of inputs. We can also define the same frontier by the minimal quantity of inputs necessary to produce a given vector of outputs. There are still other alternatives. For example, we may be satisfied to optimize on a subset of inputs. Banker and Morey (1986) used this strategy in their efficiency analysis for exogenously fixed inputs and outputs. When completely or partially fixed or out of the manager's control, an input is said to be quasi-fixed or nondiscretionary.

The quasi-fixed inputs, such as the size of buildings or certain large and bulky pieces of equipment are fixed in the short run, but they can vary in the long run following depreciation and/or new acquisitions. This aspect of the quasi-fixed inputs should be modeled. Of course, investment is an input (possibly a vector), which must be taken into account in the definition of technology. The impact of investment on the production level is known as adjustment costs. The temporary (or one-period) production function depends on both investment and variable inputs (under the immediate control of the manager) and quasi-fixed inputs (the capital).

Secondly, we define a variable cost function. Hence, we have two representations of technology: the temporary production frontier and the variable cost frontier. Interesting links can be established between the two. Except for the introduction of investment and adjustment costs, our DEA model is simply a modification of the model of Banker and Morey (1986).

However, this modification is not minor because it includes the analysis of the impact of the current decisions on the firm's future prospects. To invest implies that future outputs will increase and future costs and profits will be affected. Unless we were ready to deal with a shortsighted DMU, the future impacts ought to be built-in in the current decision-making.

The choice of intertemporal decision rule is itself an interesting subject of research especially in the public sector. Adding an intertemporal adjustment constraint into the firm's discounted cumulative cost minimization problem, we derive the relation between the DEA variables of the cost function and those of the temporary production function.

Finally, we also show how to recover the returns to scale and the implicit prices of both the capital and the investment. More importantly, we tackle the question of the measurement of the efficiency; both a novelty and a difficult task compared to the static case.

2. Temporary Production Frontier, Adjustment Cost and Cost of Technical Inefficiencies

Let $F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, \boldsymbol{i}) = f(\boldsymbol{y})\text{-}g(\boldsymbol{x}, \boldsymbol{k}, \boldsymbol{i}) = 0$ be the production frontier. In order to determine it, the following minimization problem must be solved for each DMU^h ($h = 1, \ldots, H$),

$$\begin{aligned} & \text{Min } \theta^h && (P) \\ s.t. \quad & f(\boldsymbol{y}^h) \le g(\theta^h \boldsymbol{x}^h, \boldsymbol{k}^h, \boldsymbol{i}^h) \\ & \theta^h > 0, \end{aligned}$$

where $\boldsymbol{y}^h \equiv (y^h_1, \ldots, y^h_M)'$ is the M-vector of outputs; $\boldsymbol{x}^h \equiv (x^h_1, \ldots, x^h_N)'$ is the N-vector of variable inputs; $\boldsymbol{k}^h \equiv (k^h_1, \ldots, k^h_L)'$ is the L-vector of quasi-fixed inputs; $\boldsymbol{i}^h \equiv (i^h_1, \ldots, i^h_L)'$ is the corresponding investment vector. We assume that adjustment costs are present, so $\partial g/\partial|\boldsymbol{i}| < \boldsymbol{0}$ for each $\boldsymbol{i} \neq \boldsymbol{0}$. θ^h, a scalar, is a measure of efficiency, and $h = 1, \ldots, H$ is a DMU index. The constraint inequality $f(\boldsymbol{y}^h) \le g(\theta^h \boldsymbol{x}^h, \boldsymbol{k}^h, \boldsymbol{i}^h)$ describes a *free disposal hull* (FDH) production possibility set. Problem (P) identifies whether the level of variable inputs can be reduced given the available

3 For clarity, we omit the time indexes for the matrices and right-hand side elements of the following linear program.

level of quasi-fixed factors and investment within the same production possibility space.

The production frontier involved in (P) can be approximated by the combination of the hyperfacets linking the "outer" DMUs. For simplicity, we assume symmetry between the investment cost and disinvestments cost so that the investment can be measured with its absolute value. The linear approximation of (P) can be computed using H linear programs as follows: [3]

$$TE^h = \underset{\theta^h,\lambda}{Min}\ \theta^h \qquad (TE\text{-}primal)$$

$$\begin{aligned} \text{s.t.}\quad & \boldsymbol{Y}'\boldsymbol{\lambda} \geq \boldsymbol{y}^h && (\boldsymbol{\alpha}_y) \\ & |\boldsymbol{I}|'\boldsymbol{\lambda} \geq |\boldsymbol{i}^h| && (\boldsymbol{\alpha}_i) \\ & \boldsymbol{X}'\boldsymbol{\lambda} \leq \theta^h \boldsymbol{x}^h && (\boldsymbol{\alpha}_x) \\ & \boldsymbol{K}'\boldsymbol{\lambda} \leq \boldsymbol{k}^h && (\boldsymbol{\alpha}_k) \\ & \mathbf{1}'\boldsymbol{\lambda} = 1 && (\alpha_c) \\ & \boldsymbol{\lambda} \geq \mathbf{0}, \end{aligned}$$

where TE^h stands for *technical efficiency*, the α's are corresponding associated variables and

$$\boldsymbol{Y} = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1M} \\ y_{21} & y_{22} & \dots & y_{2M} \\ \dots & \dots & \dots & \dots \\ y_{H1} & y_{H2} & \dots & y_{HM} \end{pmatrix} = \begin{pmatrix} \boldsymbol{y}^{1\prime} \\ \boldsymbol{y}^{2\prime} \\ \dots \\ \boldsymbol{y}^{H\prime} \end{pmatrix}$$

is the output matrix of the H DMUs of dimension $H{\times}M$; similarly, $\boldsymbol{X}$ is the variable input matrix of dimension $H{\times}N$

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1N} \\ x_{21} & x_{22} & \dots & x_{2N} \\ \dots & \dots & \dots & \dots \\ x_{H1} & x_{H2} & \dots & x_{HN} \end{pmatrix} = \begin{pmatrix} \boldsymbol{x}^{1\prime} \\ \boldsymbol{x}^{2\prime} \\ \dots \\ \boldsymbol{x}^{H\prime} \end{pmatrix};$$

$\boldsymbol{K}$ is the quasi-fixed inputs matrix of dimension $H{\times}L$

$$\boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1L} \\ k_{21} & k_{22} & \dots & k_{2L} \\ \dots & \dots & \dots & \dots \\ k_{H1} & k_{H2} & \dots & k_{HL} \end{pmatrix} = \begin{pmatrix} \boldsymbol{k}^{1\prime} \\ \boldsymbol{k}^{2\prime} \\ \dots \\ \boldsymbol{k}^{H\prime} \end{pmatrix};$$

$|\boldsymbol{I}|$ is the $H{\times}L$ matrix of the absolute value of investment-disinvestment

$$|\boldsymbol{I}| = \begin{pmatrix} |i_{11}| & |i_{12}| & \dots & |i_{1L}| \\ |i_{21}| & |i_{22}| & \dots & |i_{2L}| \\ \dots & \dots & \dots & \dots \\ |i_{H1}| & |i_{H2}| & \dots & |i_{HL}| \end{pmatrix} = \begin{pmatrix} |\boldsymbol{i}^1|' \\ |\boldsymbol{i}^2|' \\ \dots \\ |\boldsymbol{i}^H|' \end{pmatrix};$$

and $\mathbf{1}$ is a vector of ones with appropriate dimension. If not mentioned, the dimension of vector

1 is equal to the number of DMU. The $\boldsymbol{\alpha}$'s are the Lagrange multipliers associated with the corresponding constraints. All the DMUs on the final frontiers will be characterized by their optimal value of θ=1. The first four groups of restrictions in *TE-primal* constitute the DEA form of the FDH production possibility set. The fifth one is a convexity restriction.

The Lagrange function of the problem *TE-primal* can be written as

$$\begin{aligned} L_{TE} &= \theta^h + \boldsymbol{\alpha}_y'(\boldsymbol{y}^h - \boldsymbol{Y}'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i'(|\boldsymbol{i}^h| - |\boldsymbol{I}|'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_x'(\theta^h \boldsymbol{x}^h - \boldsymbol{X}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_k'(\boldsymbol{k}^h - \boldsymbol{K}'\boldsymbol{\lambda}) + \alpha_c(\mathbf{1}'\boldsymbol{\lambda} - 1) \\ &= (\boldsymbol{\alpha}_y'\boldsymbol{y}^h + \boldsymbol{\alpha}_i'|\boldsymbol{i}^h| - \boldsymbol{\alpha}_k'\boldsymbol{k}^h - \alpha_c) + \theta^h(1 - \boldsymbol{\alpha}_x'\boldsymbol{x}^h) - [\boldsymbol{\alpha}_y'\boldsymbol{Y}' + \boldsymbol{\alpha}_i'|\boldsymbol{I}|' - \boldsymbol{\alpha}_x'\boldsymbol{X}' - \boldsymbol{\alpha}_k'\boldsymbol{K}' - \alpha_c\mathbf{1}']\boldsymbol{\lambda}. \end{aligned}$$

At this stage, it is worthwhile to highlight our assumption about the symmetry between investment and disinvestment costs. Without this hypothesis, the constraint in (*P*) takes its original form of two inequalities, $\boldsymbol{I}_+'\boldsymbol{\lambda} \geq \boldsymbol{i}^h_+$ and $\boldsymbol{I}_-'\boldsymbol{\lambda} \geq \boldsymbol{i}^h_-$, where $\boldsymbol{i}^h_+ \in \boldsymbol{R}^L_+$ is the investment vector, $\boldsymbol{i}^h_- \in \boldsymbol{R}^L_+$ is the absolute value of the disinvestment vector of the DMU^h, $\boldsymbol{R}^L_+$ is the subset of the vectors with non-negative components in the L-dimension Euclidian space, and $\boldsymbol{I}_+$ and $\boldsymbol{I}_-$ are the matrices of, respectively, investment and disinvestment of all DMU's. So, $\boldsymbol{i}^h = \boldsymbol{i}^h_+ - \boldsymbol{i}^h_-$ and $\boldsymbol{I} = \boldsymbol{I}_+ - \boldsymbol{I}_-$, $|\boldsymbol{i}^h| = \boldsymbol{i}^h_+ + \boldsymbol{i}^h_-$ and $|\boldsymbol{I}| = \boldsymbol{I}_+ + \boldsymbol{I}_-$. Without the symmetry assumption, the Lagrange function of (*P*) is

$$\mathcal{L}_{TE} = \theta^h + \boldsymbol{\alpha}_y'(\boldsymbol{y}^h - \boldsymbol{Y}'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i^{+\prime}(\boldsymbol{i}^h_+ - \boldsymbol{I}_+'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i^{-\prime}(\boldsymbol{i}^h_- - \boldsymbol{I}_-'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_x'(\theta^h \boldsymbol{x}^h - \boldsymbol{X}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_k'(\boldsymbol{k}^h - \boldsymbol{K}'\boldsymbol{\lambda}) + \alpha_c(\mathbf{1}'\boldsymbol{\lambda} - 1).$$

The symmetry between the investment cost and disinvestment cost means that $\partial g/\partial \boldsymbol{i}^h_+ = \partial g/\partial \boldsymbol{i}^h_-$ for the marginal adjustment costs measured in terms of outputs. Formally, $\partial g(\boldsymbol{i}^0)/\partial \boldsymbol{i} = -\partial g(-\boldsymbol{i}^0)/\partial \boldsymbol{i}$ for any $\boldsymbol{i}^0$. This corresponds to the equality $\boldsymbol{\alpha}_i^+ = \boldsymbol{\alpha}_i^-$ whenever we use *TE-primal* as the linear approximate of (*P*). Let $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_i^+ = \boldsymbol{\alpha}_i^-$. Then

$$\begin{aligned} \mathcal{L}_{TE} &= \theta^h + \boldsymbol{\alpha}_y'(\boldsymbol{y}^h - \boldsymbol{Y}'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i'(\boldsymbol{i}^h_+ + \boldsymbol{i}^h_- - \boldsymbol{I}_+'\boldsymbol{\lambda} - \boldsymbol{I}_-'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_x'(\theta^h \boldsymbol{x}^h - \boldsymbol{X}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_k'(\boldsymbol{k}^h - \boldsymbol{K}'\boldsymbol{\lambda}) + \alpha_c(\mathbf{1}'\boldsymbol{\lambda} - 1) \\ &= \theta^h + \boldsymbol{\alpha}_y'(\boldsymbol{y}^h - \boldsymbol{Y}'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i'(|\boldsymbol{i}^h| - |\boldsymbol{I}|'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_x'(\theta^h \boldsymbol{x}^h - \boldsymbol{X}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_k'(\boldsymbol{k}^h - \boldsymbol{K}'\boldsymbol{\lambda}) + \alpha_c(\mathbf{1}'\boldsymbol{\lambda} - 1) \\ &= L_{TE}\,. \end{aligned}$$

The assumption of symmetry between the investment and disinvestment costs simplifies the writings of our linear approximation model, but it is not essential. One may re-decompose $\boldsymbol{i}^h$ and

$\boldsymbol{I}$ into investments and disinvestments whenever needed.

The dual problem of (*TE-primal*) is

$$\text{Max}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}_y'\boldsymbol{y}^h + \boldsymbol{\alpha}_i'|\boldsymbol{i}^h| - \boldsymbol{\alpha}_k'\boldsymbol{k}^h - \alpha_c) \qquad (\textit{TE-dual})$$

$$\begin{aligned} \text{s.t.} \quad & \boldsymbol{\alpha}_x'\boldsymbol{x}^h = 1 & (\theta^h) \\ & \boldsymbol{Y}\boldsymbol{\alpha}_y + |\boldsymbol{I}|\,\boldsymbol{\alpha}_i - \boldsymbol{X}\boldsymbol{\alpha}_x - \boldsymbol{K}\boldsymbol{\alpha}_k - \alpha_c\mathbf{1} \le \mathbf{0} & (\boldsymbol{\lambda}) \\ & \boldsymbol{\alpha}_y,\ \boldsymbol{\alpha}_i,\ \boldsymbol{\alpha}_x,\ \boldsymbol{\alpha}_k \ \ge \mathbf{0} & \\ & \alpha_c \text{ free.} & \end{aligned}$$

In standard notations, the technology should be written as $F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, \boldsymbol{i}) = 0$, especially in the absence of symmetry of adjustment costs. However, for the linear approximation of the technology to be comparable with our DEA model with symmetric adjustment costs, we use $F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, |\boldsymbol{i}|) = 0$ rather than $F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, \boldsymbol{i}) = 0$.

At a frontier point $\boldsymbol{O} = (\boldsymbol{y}^0, \boldsymbol{x}^0, |\boldsymbol{i}^0|, \boldsymbol{k}^0)$, the first-order Taylor expansion is

$$\begin{aligned} F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, |\boldsymbol{i}|) \approx\; & F(\boldsymbol{y}^0, \boldsymbol{x}^0, \boldsymbol{k}^0, |\boldsymbol{i}^0|) + F_{\boldsymbol{y}}(\boldsymbol{y}^0, \boldsymbol{x}^0, \boldsymbol{k}^0, |\boldsymbol{i}^0|)\,(\boldsymbol{y} - \boldsymbol{y}^0) \\ & + F_{\boldsymbol{x}}(\boldsymbol{y}^0, \boldsymbol{x}^0, \boldsymbol{k}^0, |\boldsymbol{i}^0|)\,(\boldsymbol{x} - \boldsymbol{x}^0) + F_{\boldsymbol{k}}(\boldsymbol{y}^0, \boldsymbol{x}^0, \boldsymbol{k}^0, |\boldsymbol{i}^0|)\,(\boldsymbol{k} - \boldsymbol{k}^0) \\ & + F_{|\boldsymbol{i}|}(\boldsymbol{y}^0, \boldsymbol{x}^0, \boldsymbol{k}^0, |\boldsymbol{i}^0|)\,(|\boldsymbol{i}| - |\boldsymbol{i}^0|), \end{aligned}$$

where the partial derivatives $F_{\boldsymbol{y}}$, $F_{\boldsymbol{x}}$, $F_{\boldsymbol{k}}$, and $F_{|\boldsymbol{i}|}$ are row vectors; $(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, |\boldsymbol{i}|)$ is an arbitrary point in an appropriate neighborhood of $\boldsymbol{O}$. Of course, the smoothness of $F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, |\boldsymbol{i}|)$ is required throughout the paper, as it is usually assumed in the literature.

On the other hand, the Lagrange function of the problem (P) is, for DMU^h,

$$\begin{aligned} L^h = &\ \theta^h + \mu F(\boldsymbol{y}^h, \theta^h\boldsymbol{x}^h, \boldsymbol{k}^h, |\boldsymbol{i}^h|) \\ \approx &\ \theta^h + \mu[F(\boldsymbol{y}^0, \boldsymbol{x}^0, |\boldsymbol{i}^0|, \boldsymbol{k}^0) + \boldsymbol{F}_{\boldsymbol{y}}(\boldsymbol{y}^h - \boldsymbol{y}^0) + \boldsymbol{F}_{\boldsymbol{x}}(\theta^h\boldsymbol{x}^h - \boldsymbol{x}^0) + \boldsymbol{F}_{\boldsymbol{k}}(\boldsymbol{k}^h - \boldsymbol{k}^0) + \boldsymbol{F}_{|\boldsymbol{i}|}(|\boldsymbol{i}^h| - |\boldsymbol{i}^0|)]\ . \end{aligned}$$

Note that $L_{TE} \approx L^h$, *i.e.*

$$\begin{aligned} & \theta^h + \alpha_c(\mathbf{1}'\boldsymbol{\lambda} - 1) + \boldsymbol{\alpha}_y'(\boldsymbol{y}^h - \boldsymbol{Y}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_x'(\theta^h\boldsymbol{x}^h - \boldsymbol{X}'\boldsymbol{\lambda}) - \boldsymbol{\alpha}_k'(\boldsymbol{k}^h - \boldsymbol{K}'\boldsymbol{\lambda}) + \boldsymbol{\alpha}_i'(|\boldsymbol{i}^h| - |\boldsymbol{I}|'\boldsymbol{\lambda}) \\ & \approx \theta^h + \mu F(\boldsymbol{y}^0, \boldsymbol{x}^0, |\boldsymbol{i}^0|, \boldsymbol{k}^0) + \mu\boldsymbol{F}_{\boldsymbol{y}}(\boldsymbol{y}^h - \boldsymbol{y}^0) + \mu\boldsymbol{F}_{\boldsymbol{x}}(\theta^h\boldsymbol{x}^h - \boldsymbol{x}^0) + \mu\boldsymbol{F}_{\boldsymbol{k}}(\boldsymbol{k}^h - \boldsymbol{k}^0) + \mu\boldsymbol{F}_{|\boldsymbol{i}|}(|\boldsymbol{i}^h| - |\boldsymbol{i}^0|). \qquad (1) \end{aligned}$$

To ensure a good approximation in (1), we need to carefully choose a frontier point $\boldsymbol{O}$ for the given DMU^h. For example, $\boldsymbol{O}$ may be taken as $(\boldsymbol{y}^h, \psi^h\boldsymbol{x}^h, |\boldsymbol{i}^h|, \boldsymbol{k}^h)$ with $0 < \psi^h \le \theta^h$, for any given DMU^h. Let us highlight this point with the help of Figure 1.

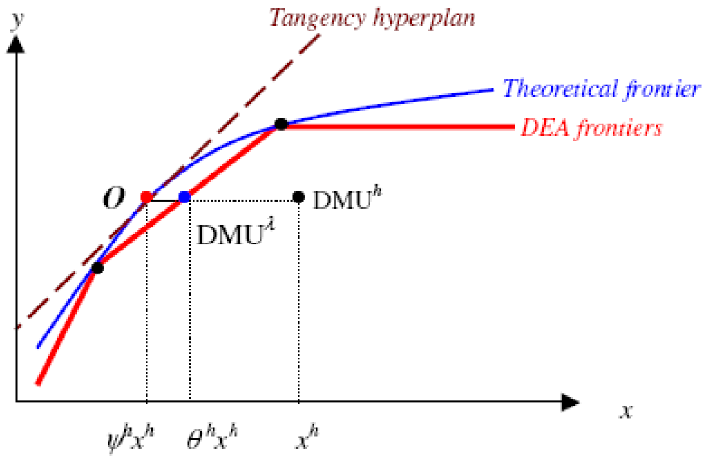

Figure 1 Production frontiers and their linear approximation

The tangency hyperplan at ***O*** and the small facet are approximately parallel. The gradients of the facet at the *fictive* point DMU$^{\lambda}$ and that of the theoretical frontier surface at ***O*** should be considered equal. If DMU$^{\lambda}$ is an extreme point, the α's are not unique. However, various values of α's do not greatly differ when the number of the observed DMU's are sufficiently great, because we have assumed the smoothness of *F*. Furthermore, under the assumption of second order differentiability of the function *F*, one can evaluate the partial derivatives of *F* at ***O*** by those correspondingly at DMU$^{\lambda}$. The error incurred by this approximation is at most $(\theta - \psi)\|x\|$, where $\|\bullet\|$ is the Euclidian norm. Hence, (1) yields:

$$\alpha_y = \mu F^0{}_y{}', \ \alpha_i = \mu F^0{}_{|i|}{}', \ \alpha_x = -\mu F^0{}_{(\theta x)}{}', \quad \alpha_k = -\mu F^0{}_k{}',$$

if we identify "=" with "≈". On the other hand, the first-order conditions of (*P*) give $\partial L^h/\partial \theta^h = 0$, *i.e.*, $1 + \mu F_{(\theta x)} x^h = 0$ or $\mu = -1/(F_{(\theta x)} x^h)$. Therefore, the links between the technology and the dual variables of technical efficiency problem are

$$\alpha_y = -F_y{}'/(F_{(\theta x)} x^h), \alpha_i = -F_{|i|}{}'/(F_{(\theta x)} x^h), \ \alpha_x = F_{(\theta x)}{}'/(F_{(\theta x)} x^h), \ \alpha_k = F_k{}'/(F_{(\theta x)} x^h).$$

Further, from $F_{(\theta x)}{}' = \alpha_x (F_{(\theta x)} x^h)$, it follows that $F_x \mathbf{1}_N = (\mathbf{1}_N{}' \alpha_x)(F_x x^h)$, which is valid at any frontier point[4]. We can now rewrite these relations between the partial derivatives and the α's.

[4] $\mathbf{1}_N$ is a *N*-vector of ones; *N* being the number of variable inputs.

$$\left.\begin{aligned} \boldsymbol{F}_y/(\boldsymbol{F}_x\mathbf{1}_N) &= -\,\boldsymbol{\alpha}_y'/(\mathbf{1}_N'\boldsymbol{\alpha}_x),\\ \boldsymbol{F}_{|i|}/(\boldsymbol{F}_x\mathbf{1}_N) &= -\,\boldsymbol{\alpha}_i'/(\mathbf{1}_N'\boldsymbol{\alpha}_x),\\ \boldsymbol{F}_x/(\boldsymbol{F}_x\mathbf{1}_N) &= \boldsymbol{\alpha}_x'/(\mathbf{1}_N'\boldsymbol{\alpha}_x),\\ \boldsymbol{F}_k/(\boldsymbol{F}_x\mathbf{1}_N) &= \boldsymbol{\alpha}_k'/(\mathbf{1}_N'\boldsymbol{\alpha}_x). \end{aligned}\right\} \qquad (2)$$

The measure of technical efficiency can be expressed in terms of variable input levels. The cost of technical inefficiency is measured by $(1-\theta^h)C^h_{obs} = (1-\theta^h)\,\boldsymbol{w}^{h\prime}\,\boldsymbol{x}^h$, where C^h_{obs} is the observed variable cost of DMU h and w^h_n is the market price of the nth variable input. The technically efficient cost can be calculated as $TE^h \times C^h_{obs} = \theta^h \times C^h_{obs}$.

3. The Variable Cost Function in the Presence of Adjustment Cost and the Cost of Allocative Inefficiencies

The cost of technical inefficiency results from a choice of inputs that is below the production frontier. Other costs might result because all choices on the production frontier are not equivalent. Given the price vector $\boldsymbol{w}^h$ of variable input factors $\boldsymbol{x}^h$, some points on the frontier will yield a minimal production variable cost. To determine these points, the following variable cost minimization problem must be solved:

$$C^h_{min} = \mathrm{Min}_{\boldsymbol{x}}\{\, \boldsymbol{w}^h\boldsymbol{x}^h \mid f(\boldsymbol{y}^h) \le g(\boldsymbol{x}^h, \boldsymbol{k}^h, |\boldsymbol{i}^h|)\} \qquad (C)$$

It can be approximated with the following linear program:

$$C^h_{AE} \equiv C^h(\boldsymbol{w}^h, \boldsymbol{k}^h, \boldsymbol{y}^h, |\boldsymbol{i}^h|) \qquad (AE\text{-}primal)$$

$$= \mathrm{Min}_{x_E,\mu}\; \boldsymbol{w}^{h\prime}\boldsymbol{x}_E$$

s.t.

$$\begin{aligned} \boldsymbol{Y}'\boldsymbol{\mu} &\ge \boldsymbol{y}^h && (\boldsymbol{\beta}_y)\\ |\boldsymbol{I}|'\boldsymbol{\mu} &\ge |\boldsymbol{i}^h| && (\boldsymbol{\beta}_i)\\ \boldsymbol{X}'\boldsymbol{\mu} &\le \boldsymbol{x}^h && (\boldsymbol{\beta}_x)\\ \boldsymbol{K}'\boldsymbol{\mu} &\le \boldsymbol{k}^h && (\boldsymbol{\beta}_k)\\ \boldsymbol{x}_E - \boldsymbol{X}'\boldsymbol{\mu} &= \mathbf{0} && (\boldsymbol{\beta}_E)\\ \mathbf{1}'\boldsymbol{\mu} &= 1 && (\beta_c)\\ \boldsymbol{\mu} &\ge \mathbf{0} \end{aligned}$$

where $\boldsymbol{x_E}$ is the solution to the variable cost minimization problem. The cost of allocation inefficiencies is the difference between the efficient cost calculated in the previous section and the minimal cost, that is ($\theta^h \times C^h_{obs}$ - C^h_{min}). In the traditional way, the allocative inefficiency (AE^h) expressed as a percentage of the efficient cost (C^h_{obs}) is $AE^h = C^h_{min} / (\theta^h \times C^h_{obs})$.

Added together, the allocation and technical inefficiencies yield the total cost surplus. Similarly, the product of the allocation and technical inefficiencies yields the global inefficiency as a percentage of the observed cost: $OE^h = TE^h \times AE^h = \theta^h \times AE^h = C^h_{min}/C^h$.

The Lagrange function of (*AE-primal*) is

$$L_{AE} = \boldsymbol{w^h}'\boldsymbol{x}_E + \boldsymbol{\beta_y}'(\boldsymbol{y^h} - \boldsymbol{Y}'\boldsymbol{\mu}) + \boldsymbol{\beta_i}'(|\boldsymbol{i^h}| - |\boldsymbol{I}|'\boldsymbol{\mu}) - \boldsymbol{\beta_x}'(\boldsymbol{x^h} - \boldsymbol{X}'\boldsymbol{\mu}) - \boldsymbol{\beta_k}'(\boldsymbol{k^h} - \boldsymbol{K}'\boldsymbol{\mu}) - \boldsymbol{\beta_E}'(\boldsymbol{x_E} - \boldsymbol{X}'\boldsymbol{\mu}) + \beta_c(\mathbf{1}'\boldsymbol{\mu} - 1)$$
$$= [\boldsymbol{\beta_y}'\boldsymbol{y^h} + \boldsymbol{\beta_i}'|\boldsymbol{i^h}| - \boldsymbol{\beta_x}'\boldsymbol{x^h} - \boldsymbol{\beta_k}'\boldsymbol{k^h} - \beta_c] - [\boldsymbol{\beta_y}'\boldsymbol{Y}' + \boldsymbol{\beta_i}'|\boldsymbol{I}|' - (\boldsymbol{\beta_x}' + \boldsymbol{\beta_E}')\boldsymbol{X}' - \boldsymbol{\beta_k}'\boldsymbol{K}' - \beta_c\mathbf{1}']\boldsymbol{\mu} + (\boldsymbol{w^h}' - \boldsymbol{\beta_E}')\boldsymbol{x}_E.$$

The dual problem of (*AE-primal*) is written as

$$\text{Max}_{\beta}[\boldsymbol{\beta_y}'\boldsymbol{y^h} + \boldsymbol{\beta_i}'|\boldsymbol{i}^h| - \boldsymbol{\beta_x}'\boldsymbol{x^h} - \boldsymbol{\beta_k}'\boldsymbol{k^h} - \beta_c] \qquad (AE\text{-}dual)$$
$$s.t. \qquad \boldsymbol{w^h} = \boldsymbol{\beta_E} \qquad (\boldsymbol{x}_E)$$
$$\boldsymbol{Y\beta_y} + |\boldsymbol{I}|\boldsymbol{\beta_i} - \boldsymbol{X}(\boldsymbol{\beta_x} + \boldsymbol{\beta_E}) - \boldsymbol{K\beta_k} - \beta_c\mathbf{1} \le \mathbf{0} \qquad (\boldsymbol{\mu})$$
$$\boldsymbol{\beta_y}, \boldsymbol{\beta_i}, \boldsymbol{\beta_x}, \boldsymbol{\beta_k} \ge \mathbf{0}$$
$$\beta_c\ free.$$

The Envelope Theorem shows that at the optimum,

$$\left.\begin{aligned}
\partial C^h_{AE}/\partial \boldsymbol{w^h} &= \partial L_{AE}/\partial \boldsymbol{w^h} = \boldsymbol{x}_E' \\
\partial C^h_{AE}/\partial \boldsymbol{y^h} &= \partial L_{AE}/\partial \boldsymbol{y^h} = \boldsymbol{\beta_y}' \\
\partial C^h_{AE}/\partial |\boldsymbol{i^h}| &= \partial L_{AE}/\partial |\boldsymbol{i^h}| = \boldsymbol{\beta_i}' \\
\partial C^h_{AE}/\partial \boldsymbol{k^h} &= \partial L_{AE}/\partial \boldsymbol{k^h} = -\boldsymbol{\beta_k}'
\end{aligned}\right\} \qquad (3)$$

At the optimal level of the problem (*C*), $C^h_{min} = \boldsymbol{w^h}'\boldsymbol{x} + \phi F(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{k}, |\boldsymbol{i}|)$, where ϕ is a Lagrange multiplier. By Envelope Theorem, $\partial C^h_{\min}/\partial \boldsymbol{w}^h = \boldsymbol{x}'$. From first-order conditions, $\boldsymbol{w^h}' = -\phi\partial F/\partial \boldsymbol{x}$. If the sum of the variable input prices is normalized to 1,[4] then $1 = \boldsymbol{w^h}'\mathbf{1_N} = -\phi \boldsymbol{F_x}\mathbf{1_N}$, thus $\phi = -1/\boldsymbol{F_x}\mathbf{1_N}$. Again by using Envelope Theorem on optimized C^h_{min}, we have that

4 Naturally, it would be possible to impose other types of normalization. For example, we could impose that the *n*th price be equal to 1. This implies an asymmetry in the treatment of inputs. For that reason, we preferred the above normalization. Nevertheless, it should be clear that the results must be modified accordingly to the normalization rule.

$$\left.\begin{aligned} \partial C^{h}_{min}/\partial \boldsymbol{y} &= \phi\partial F/\partial \boldsymbol{y} = -\boldsymbol{F}_y/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}};\\ \partial C^{h}_{min}/\partial\,|\boldsymbol{i}| &= \phi\partial F/\partial\,|\boldsymbol{i}| = -\boldsymbol{F}_{|i|}/\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}};\\ \partial C^{h}_{min}/\partial \boldsymbol{k} &= \phi\partial F/\partial \boldsymbol{k} = -\boldsymbol{F}_k/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}}. \end{aligned}\right\} \qquad (4)$$

Under approximation sense, $C^{h}_{AE} = C^{h}_{min}$. We can have, from (6) and (7), that

$$\left.\begin{aligned} \boldsymbol{F}_y\,/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}} &= -\ \boldsymbol{\beta}_y'\\ \boldsymbol{F}_{|i|}\,/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}} &= -\ \boldsymbol{\beta}_i'\\ \boldsymbol{F}_x\,/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}} &= \ \ \boldsymbol{w}^{h\prime}\\ \boldsymbol{F}_k\,/\,\boldsymbol{F}_x\mathbf{1}_{\mathrm{N}} &= \ \ \boldsymbol{\beta}_k' \end{aligned}\right\} \qquad (5)$$

This is an alternative way to express the first-order derivatives of the production function.

From (2) and (5), an interesting relationship between the dual variables of technical efficiency problem and allocation efficiency problem can be drawn under the normalization,

$$\boldsymbol{\beta}_y = \boldsymbol{\alpha}_y\,/\mathbf{1}_{\mathrm{N}}'\boldsymbol{\alpha}_x\ ,\ \ \boldsymbol{\beta}_i = \boldsymbol{\alpha}_i/\mathbf{1}_{\mathrm{N}}'\boldsymbol{\alpha}_x\ ,\ \ \boldsymbol{\beta}_E = \boldsymbol{w}^h = \boldsymbol{\alpha}_x\,/\mathbf{1}_{\mathrm{N}}'\boldsymbol{\alpha}_x\ ,\ \ \boldsymbol{\beta}_k = \boldsymbol{\alpha}_k\,/\mathbf{1}'\boldsymbol{\alpha}_x\ . \qquad (6)$$

4. Intertemporal Decisions

The investment decisions are the result of an optimization over several periods. For any specific firm (for simplicity, index h is omitted in this section),

$$\mathrm{Min}_{\{i_\tau\}}\ \{\ \sum_{\tau=t}^{T}\ \rho^{\tau}[C_\tau(\boldsymbol{w}_\tau, \boldsymbol{k}_\tau, \boldsymbol{i}_\tau, \boldsymbol{q}_\tau) + \boldsymbol{q}_\tau'\boldsymbol{i}_\tau]\ \}$$

$$s.t. \qquad \boldsymbol{k}_{\tau+1} = (\mathbf{E}\text{-}\boldsymbol{\Delta})\boldsymbol{k}_\tau + \boldsymbol{i}_\tau, \quad \tau = t,\ t+1,\ \ldots\ ,\ T \qquad (7)$$

where ρ is the discount factor, $\mathbf{E}$ is the ($L\times L$)-identity matrix, $\boldsymbol{\Delta}$ is the ($L\times L$)- rate-of-depreciation matrix (diagonal) which is supposed constant and defined by $\boldsymbol{\Delta}_{ll} = \delta_l$ and $\boldsymbol{\Delta}_{lv} = 0$, for $v \neq l$, and $\boldsymbol{q}_\tau$ is the investment-price vector. Serving as the control instrument, investment $\boldsymbol{i}_\tau$ may be optimally decided to be positive or negative or null. Unlike the previous sections of this chapter, it is better to let the investments take their algebraic values in the above intertemporal decision model, instead of absolute values.

The Hamilton function of (7) can be written as

$$\mathrm{H} = \rho^{\tau}[C_\tau(\boldsymbol{w}_\tau, \boldsymbol{k}_\tau, \boldsymbol{i}_\tau, \boldsymbol{q}_\tau) + \boldsymbol{q}_\tau'\boldsymbol{i}_\tau] - \boldsymbol{\varphi}_{\tau+1}'\ (\boldsymbol{i}_\tau - \boldsymbol{\Delta}\boldsymbol{k}_\tau),$$

where the Hamilton multiplier vector (of dimension L), $\boldsymbol{\varphi}_{\tau+1,}$ is already actualized. At optimum, the following necessary conditions will be verified.

$$\rho^{\tau}[\partial C_{\tau}/\partial \boldsymbol{i}_{\tau}'+\boldsymbol{q}_{\tau}] = \boldsymbol{\varphi}_{\tau+1}, \qquad \text{for } \tau = t, t+1, \ldots, T \tag{8}$$

$$\boldsymbol{\varphi}_{\tau+2} - \boldsymbol{\varphi}_{\tau+1} = \rho^{\tau+1}\partial C_{\tau+1}/\partial \boldsymbol{k}_{\tau+1}'+\boldsymbol{\Delta\varphi}_{\tau+2}, \qquad \text{for } \tau = t, t+1, \ldots, T\text{-}1. \tag{9}$$

$$\boldsymbol{k}_{\tau+1} - \boldsymbol{k}_{\tau} = \boldsymbol{i}_{\tau}^{+} - \boldsymbol{\Delta k}_{\tau}, \qquad \text{for } \tau = t, t+1, \ldots, T.$$

The terminal conditions are $\boldsymbol{\varphi}_{T+1} = \mathbf{0}_{(L\times 1)}$.

Consider now the case of $\tau = t$. From (9),

$$(\mathbf{E}\text{-}\boldsymbol{\Delta})\boldsymbol{\varphi}_{t+2} - \boldsymbol{\varphi}_{t+1} = \rho^{t+1}\partial C_{t+1}/\partial \boldsymbol{k}_{t+1}',$$

or, with the delay-operator defined by $\mathbf{L}z_t = z_{t-1}$,

$$[(\mathbf{E}\text{-}\boldsymbol{\Delta})\mathbf{L}^{-1} - \mathbf{E}]\boldsymbol{\varphi}_{t+1} = \rho^{t+1}\partial C_{t+1}/\partial \boldsymbol{k}_{t+1}' .$$

Under common assumption, the norm of $(\mathbf{E}\text{-}\boldsymbol{\Delta})\mathbf{L}^{-1}<1$. This allows us to have the following expansion according to the basic spectrum theorem of linear operators,

$$\boldsymbol{\varphi}_{t+1} = -\sum_{s=0}^{\infty} (\mathbf{E}\text{-}\boldsymbol{\Delta})^{s}\mathbf{L}^{-s}(\rho^{t+1}\partial C_{t+1}/\partial \boldsymbol{k}_{t+1}'). \tag{10}$$

(8) and (10) lead to

$$\partial C_t/\partial \boldsymbol{i}_t' = -(1/\rho^{t})\,[\sum_{s=0}^{\infty} (\mathbf{E}\text{-}\boldsymbol{\Delta})^{s}\,(\rho^{t+s+1}\partial C_{t+s+1}/\partial \boldsymbol{k}_{t+s+1}')] - \boldsymbol{q}_t .$$

Denote $1/(1+r)^{s+1} = \rho^{t+s+1}/\rho^{t}$. We get

$$\partial C_t/\partial \boldsymbol{i}_t' = -1/(1+r)\{\sum_{s=0}^{\infty} [(\mathbf{E}\text{-}\boldsymbol{\Delta})/(1+r)]^{s}\,(\partial C_{t+s+1}/\partial \boldsymbol{k}_{t+s+1}')\} - \boldsymbol{q}_t . \tag{11}$$

(11) plays the key role in the links of decisions from period to period. It shows how the investment prices, the capital shadow prices and the adjustment shadow prices are organically related. Unfortunately it seems that, in general, even if the future shadow prices were available, the sum in (11) might be too difficult to be calculated without any further assumption. We then make the following simplification assumption.

Assumption 1: $\partial C_{t+s+1}/\partial \boldsymbol{k}_{t+s+1} \equiv \partial C_t/\partial \boldsymbol{k}_t = -\boldsymbol{b}_{kt}'$, for all s.

Under this assumption, (11) can be rewritten as

$$\begin{aligned} \partial C_t/\partial \boldsymbol{i}_t' &= 1/(1+r)\{\sum_{s=0}^{\infty} [(\mathbf{E} - \boldsymbol{\Delta})/(1+r)]^s \boldsymbol{b}_{kt}\} - \boldsymbol{q}_t \\ &= 1/(1+r)\{\mathbf{E}-[(\mathbf{E}-\boldsymbol{\Delta})/(1+r)]\}^{-1}\boldsymbol{b}_{kt} - \boldsymbol{q}_t \\ &= (r\mathbf{E} + \boldsymbol{\Delta})^{-1}\boldsymbol{b}_{kt} - \boldsymbol{q}_t \end{aligned}$$

For the lth component of $\partial C_t/\partial \boldsymbol{i}_t$, we get

$$\partial C_\tau/\partial i_{l\tau}' = b_{kl,t}/(r + \delta_l) - q_{lt}. \tag{12}$$

This is just what we have sought: the intertemporal adjustment restriction describing the relationship between the capital's shadow prices and its marginal adjustment costs.

Until now, $i_{l\tau}$ takes its algebraic value in the intertemporal decision model. To incorporate this dynamic relationship into DEA models where we use the absolute value of the investments, we must find the partial derivates of the variable cost function *w. r. t.* $|\boldsymbol{i}_t|$. $\partial C_t/\partial i_{lt}$ has the same sign with i_{lt} for every l =1, 2, ..., L. Actually according to (12), the firm invests in lth quasi-fixed input (i_{lt} >0) when $b_{kl,t}/(r+\delta_l)-q_{lt}>0$; it disinvests in lth quasi-fixed input (i_{lt} <0) when $b_{kl,t}/(r+\delta_l)-q_{lt}<0$ and it does not change lth quasi-fixed input (i_{lt} =0) when $b_{kl,t}/(r+\delta_l) - q_{lt}=0$.

Then it is always true that for each l = 1, 2, …, L,

$$\partial C_t/\partial |i_{lt}|' = |b_{kl,t}/(r+\delta_l)-q_{lt}|,$$

that is,

$$\partial C_t/\partial |\boldsymbol{i}_t|' = \begin{pmatrix} |b_{k1,t}/(r+\delta_1)-q_{1t}| \\ |b_{k2,t}/(r+\delta_2)-q_{2t}| \\ \cdots\cdots\cdots \\ |b_{kL,t}/(r+\delta_L)-q_{Lt}| \end{pmatrix}. \tag{13}$$

5. Dynamic DEA Model

Now it is possible to incorporate the intertemporal adjustment restrictions we have obtained in (13) on our DEA models (*TE-primal* or *dual*) and (*AE-primal* or *dual*). Without these intertemporal restrictions, the decision maker can independently make the production plan for each period. The only factor that influences the current investment quantity is the current purchase price of the quasi-fixed inputs. But with the restrictions (16), he should take the impacts of future capitals' shadow prices into account, which affect the choice of current investment via the marginal adjustment costs.

5.1. Technical Efficiency

Under Assumption 1: $\partial C_{t+s+1}/\partial \boldsymbol{k}_{t+s+1}' \equiv -\boldsymbol{b}_{kt} = \boldsymbol{\beta}_{kt}$. In terms of DEA variables,

$$\boldsymbol{\beta}_{k(t+s)} \equiv \boldsymbol{\beta}_{kt}, \qquad \text{for all } s \geq 0. \tag{14}$$

Another "indirect" impact of this assumption comes from (13) and (14) and can be written as follows. For each l, l =1, 2, …, L, and s, s =0,1, 2, …

$$\beta_{il\,t+s} = \partial C_{t+s}/\partial \left| i_{lt+s} \right|' = \left| (r+\delta_l)^{-1} b_{klt} - q_{lt} \right| . \tag{15}$$

Assumption 1 also has impacts on $\boldsymbol{\alpha}$'s. This may be seen from (6) and (15). In fact, we can easily show, using the normalization $\mathbf{1}_N'\boldsymbol{w}^h = 1$, that

$$\boldsymbol{\alpha}_{k(t+s)}/\mathbf{1}_N'\boldsymbol{\alpha}_{x(t+s)} = \boldsymbol{\beta}_{k(t+s)} = \boldsymbol{\beta}_{kt} = \boldsymbol{\alpha}_{kt}/\mathbf{1}_N'\boldsymbol{\alpha}_{xt}$$
$$\boldsymbol{\alpha}_{k(t+s)} = (\mathbf{1}_N'\boldsymbol{\alpha}_{x(t+s)}/\ \mathbf{1}_N'\boldsymbol{\alpha}_{xt})\ \boldsymbol{\alpha}_{kt}, \qquad \text{for all } s \geq 0. \tag{16}$$

And for each l, l=1, 2, …, L,

$$\alpha_{ilt}/\mathbf{1}_N'\boldsymbol{\alpha}_{xt} = \beta_{ilt} = \left| (r+\delta_l)^{-1} b_{klt} - q_{lt} \right|$$
$$\alpha_{ilt} = \left| \mathbf{1}_N'\boldsymbol{\alpha}_{xt}(r+\delta_l)^{-1} b_{klt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} q_{lt} \right| = \left| (r+\delta_l)^{-1} \alpha_{klt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} q_{lt} \right| . \tag{17}$$

It is convenient to write (17) in vector form in order to rewrite our DEA models. We take off the absolute by using the sign function.

$$\alpha_{ilt} = [(r+\delta_l)^{-1} \alpha_{klt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} q_{lt}] \cdot \text{sign}[(r+\delta_l)^{-1} \alpha_{klt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} q_{lt}].$$

Thus,

$$\boldsymbol{\alpha}_{it} = \mathbf{Z}_t[(r\mathbf{E}+\boldsymbol{\Delta})^{-1} \boldsymbol{\alpha}_{kt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} \boldsymbol{q}_t], \tag{18}$$

where

$$\mathbf{Z}_t = \begin{pmatrix} \text{sign}(i_{1t}) & & O \\ & \ddots & \\ O & & \text{sign}(i_{Lt}) \end{pmatrix} .$$

since $\text{sign}((r+\delta_l)^{-1} \alpha_{klt} - \mathbf{1}_N'\boldsymbol{\alpha}_{xt} q_{lt}) = \text{sign}(i_{lt})$ for every l =1, 2, ..., L. We have $\boldsymbol{i}_t = \mathbf{Z}_t \left| \boldsymbol{i}_t \right|$. From (17) and (18), the lth component of $\alpha_{i\,(t+s)}$ is written as

$$\alpha_{il\,(t+s)} = \left| (r+\delta_l)^{-1}(\mathbf{1}_N'\alpha_{x\,(t+s)}/\,\mathbf{1}_N'\alpha_{xt})\alpha_{klt} - \mathbf{1}_N'\alpha_x\, q_{(t+s)} \right|, \quad \text{for all } s \geq 0.$$

The *TE-dual* problem in section I can be written in matrix form as

$$TE^h = \underset{\alpha}{\text{Max}}[\alpha_y' \;\; \alpha_i' \;\; \alpha_x' \;\; \alpha_k'] \begin{pmatrix} y^h \\ |i^h| \\ \mathbf{0} \\ -k^h \end{pmatrix} \qquad (TE\text{-}dual)$$

s.t.

$$\begin{pmatrix} \mathbf{0}' & \mathbf{0}' & x^{h\prime} & \mathbf{0}' \\ Y & |I| & -X & -K \end{pmatrix} \begin{pmatrix} \alpha_y \\ \alpha_i \\ \alpha_x \\ \alpha_k \end{pmatrix} \leq \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} . \qquad \begin{matrix} (\theta^h) \\ (\lambda^h) \end{matrix}$$

Substituting the above restriction in this problem we get, for the period t and for each DMU^h,

$$TE^h = \text{Max}\,[\alpha_{yt}' \;\; \alpha_{xt}' \;\; \alpha_{kt}'] \begin{pmatrix} y^h_t \\ -q^h_t{}'\mathbf{1}_N\mathbf{Z}^h_t{}'\,|i^h_t| \\ [(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^h_t{}'\,|i^h_t| - k^h \end{pmatrix} \qquad (TE\text{-}dual_{restricted})$$

s.t.

$$\begin{pmatrix} \mathbf{0}' & x^h_t{}' & \mathbf{0}' \\ Y_t & -(|I_t|\,\mathbf{Z}^h_t q^h_t \mathbf{1}_N' + X_t) & [\,|I_t|\,\mathbf{Z}^h_t\,(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1} - K_t] \end{pmatrix} \begin{pmatrix} \alpha_{yt} \\ \alpha_{xt} \\ \alpha_{kt} \end{pmatrix} \leq \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} . \qquad \begin{matrix} (\theta^h_t) \\ (\lambda^h_t) \end{matrix}$$

We can now rewrite the primal problem as follows:

$$TE^h = \underset{\theta^h_t,\, \lambda^h_t}{\text{Min}}\; [1 \quad \mathbf{0}'] \begin{pmatrix} \theta^h_t \\ \lambda^h_t \end{pmatrix} \qquad (TE\text{-}primal_{restricted})$$

s.t.

$$\begin{pmatrix} \mathbf{0} & Y_t' \\ x^h_t & -(|I_t|\,\mathbf{Z}^h_t q^h_t \mathbf{1}_N' + X_t)' \\ \mathbf{0} & [\,|I_t|\,\mathbf{Z}^h_t(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1} - K_t]' \\ 0 & \mathbf{1}' \end{pmatrix} \begin{pmatrix} \theta^h_t \\ \lambda^h_t \end{pmatrix} \geq \begin{pmatrix} y^h_t \\ -\mathbf{1}_N\, q^h_t{}'\mathbf{Z}^h_t{}'\,|i^h_t| \\ [(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^h_t{}'\,|i^h_t| - k^h_t] \\ 1 \end{pmatrix}$$

5.2. Allocative Efficiency

Similar to what we have done with *TE* model, we can now incorporate the intertemporal constraints into the *AE* problem. The direct way to do this is also to start with rewriting of *AE-dual* problem, as it is the dual variables that carry the explicit intertemporal information. Note that from (6) and (18), $\beta_{it} = \mathbf{Z}^h_t\,[(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\beta_{kt} - q^h_t]$. Put this relation into the *AE-dual* model:

$$C_{AE} = \max_{\beta}[\boldsymbol{\beta}_{yt}'\ \boldsymbol{\beta}_{xt}'\ \boldsymbol{\beta}_{kt}'\ \boldsymbol{\beta}_{Et}'\ \beta_{0t}] \begin{pmatrix} \boldsymbol{y}_t^h \\ -\boldsymbol{x}_t^h \\ [(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^h_t \,|\, \boldsymbol{i}_t^h \,|\, -\boldsymbol{k}_t^h] \\ \mathbf{0} \\ 1 \end{pmatrix} - \boldsymbol{q}^h_t{}'\mathbf{Z}^h_t \,|\, \boldsymbol{i}_t^h \,|\,. \quad (AE\text{-}dual_{restricted})$$

s.t.

$$\begin{pmatrix} \mathbf{0}' & \mathbf{0}' & \mathbf{0}' & \mathbf{E} & \mathbf{0}' \\ \boldsymbol{Y}_t & -\boldsymbol{X}_t & [\,|\boldsymbol{I}_t|\,\mathbf{Z}^h_t(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}-\boldsymbol{K}_t] & -\boldsymbol{X}_t & \mathbf{1} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_{yt} \\ \boldsymbol{\beta}_{xt} \\ \boldsymbol{\beta}_{kt} \\ \boldsymbol{\beta}_{Et} \\ \beta_{0t} \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{w}^h_t \\ |\boldsymbol{I}_t|\,\mathbf{Z}^h_t\boldsymbol{q}^h_t \end{pmatrix}. \quad \begin{matrix} (\boldsymbol{x}_E{}^h{}_t) \\ (\boldsymbol{\lambda}^h{}_t) \end{matrix}$$

Hence the primal problem becomes

$$C_{AE} = \min_{\boldsymbol{x}_E,\, \boldsymbol{\lambda}} [\boldsymbol{w}^h_t{}' \quad (\,|\boldsymbol{I}_t|\,\mathbf{Z}^h_t\boldsymbol{q}^h_t)\,'\,] \begin{pmatrix} \boldsymbol{x}_{Et} \\ \boldsymbol{\lambda}_t \end{pmatrix} -\boldsymbol{q}^h_t{}'\mathbf{Z}^h_t \,|\, \boldsymbol{i}_t^h \,|\,. \qquad (AE\text{-}primal_{restricted})$$

$$s.t. \quad \begin{pmatrix} \mathbf{0} & \boldsymbol{Y}_t' \\ \mathbf{0} & -\boldsymbol{X}_t' \\ \mathbf{0} & [\,|\boldsymbol{I}_t|\,\mathbf{Z}^h_t\,(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}-\boldsymbol{K}_t]\,' \\ \mathbf{E} & -\boldsymbol{X}_t' \\ \mathbf{0} & \mathbf{1}' \\ \mathbf{0} & \mathbf{E} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_E{}^h{}_t \\ \boldsymbol{\lambda}^h{}_t \end{pmatrix} \geq \begin{pmatrix} \boldsymbol{y}^h{}_t \\ -\boldsymbol{x}^h{}_t \\ [(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^h_t\,|\,\boldsymbol{i}^h{}_t\,|\,-\boldsymbol{k}^h{}_t] \\ \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{pmatrix}.$$

6. Results

6.1. Final Version of the Dynamic DEA Model

For the sake of clarity, we add from now on the DMU's full index h and t for all the DEA variables, as well as for matrix $\mathbf{Z}$ and vector $\boldsymbol{q}$. Now let us present the final form of the technical and allocative models in both primal and dual settings. In these models, the intertemporal constraints are introduced.

The technical efficiency primal problem is

$$TE^h = Min_{\theta,\lambda}\theta_t^h$$

$$\begin{aligned} s.t. \quad & \boldsymbol{Y}_t'\boldsymbol{\lambda}^h{}_t \geq \boldsymbol{y}^h{}_t & (\boldsymbol{\alpha}^h{}_{yt}) & \\ & \boldsymbol{x}^h{}_t\theta^h{}_t-(\,|\boldsymbol{I}_t|\,\mathbf{Z}^h{}_t\boldsymbol{q}^h{}_t\mathbf{1}_N{}'+\boldsymbol{X}_t)\,'\boldsymbol{\lambda}^h{}_t \geq -\mathbf{1}_N\boldsymbol{q}^h{}_t{}'\mathbf{Z}^h{}_t{}'\,|\,\boldsymbol{i}^h{}_t\,| & (\boldsymbol{\alpha}^h{}_{xt}) & \quad (19) \\ & \boldsymbol{G}_t'\,\boldsymbol{\lambda}^h{}_t \geq \boldsymbol{g}^h{}_t & (\boldsymbol{\alpha}^h{}_{kt}) & \quad (20) \\ & \mathbf{1}'\boldsymbol{\lambda}^h{}_t \geq 1 & & \\ & \boldsymbol{\lambda}^h{}_t \geq 0 & & \end{aligned}$$

where $\mathbf{Z}^{h}{}_{t}'|\boldsymbol{i}^{h}{}_{t}| = \boldsymbol{i}^{h}{}_{t}$ and $|\boldsymbol{I}_t|\mathbf{Z}^{h}{}_{t}$ is the *h-oriented* investment matrix, that is, the sign of $i^{h}{}_{lt}$, *i.e.*, sign($i^{h}{}_{lt}$) is imposed upon the lth quasi-fixed input of every DMU, l=1, 2, …, L; $\boldsymbol{G}^{h}{}_{t}=[\,|\boldsymbol{I}_t|\mathbf{Z}^{h}{}_{t}(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}-\boldsymbol{K}_t]$ and $\boldsymbol{g}^{h}{}_{t}=[(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^{h}{}_{t}|\boldsymbol{i}^{h}{}_{t}|-\boldsymbol{k}^{h}{}_{t}]$.

The *TE*-dual model is

$$TE^{\boldsymbol{h}} = \max\{\boldsymbol{\alpha}^{h}{}_{yt}'\boldsymbol{y}^{\boldsymbol{h}}{}_{t} - \boldsymbol{\alpha}^{h}{}_{xt}'\mathbf{1}_{\mathrm{N}}\boldsymbol{q}^{h}{}_{t}'\mathbf{Z}^{\boldsymbol{h}}{}_{t}'|\boldsymbol{i}^{\boldsymbol{h}}{}_{t}| + \boldsymbol{\alpha}^{h}{}_{kt}'\boldsymbol{g}^{\boldsymbol{h}}{}_{t}\}$$

$$s.t. \qquad \boldsymbol{x}^{\boldsymbol{h}}{}_{t}'\boldsymbol{\alpha}^{h}{}_{xt} = 1 \qquad (\theta^{h}{}_{t})$$

$$\boldsymbol{Y}_t\,\boldsymbol{\alpha}^{h}{}_{yt} - (|\boldsymbol{I}_t|\mathbf{Z}^{\boldsymbol{h}}{}_{\boldsymbol{t}}\boldsymbol{q}^{\boldsymbol{h}}{}_{t}\mathbf{1}_{\mathrm{N}}' + X_t)\boldsymbol{\alpha}^{h}{}_{xt} + \boldsymbol{G}^{\boldsymbol{h}}{}_{\boldsymbol{t}}\boldsymbol{\alpha}^{h}{}_{kt} \leq \mathbf{0}. \qquad (\lambda^{h}{}_{t})$$

The *AE*-primal problem can now be rewritten as

$$C_{AE}+\boldsymbol{q}^{\boldsymbol{h}}{}_{t}'\mathbf{Z}^{\boldsymbol{h}}{}_{t}|\boldsymbol{i}_t^{h}| = \min_{\boldsymbol{x}_E,\lambda}[\boldsymbol{w}^{\boldsymbol{h}}{}_{t}'\boldsymbol{x}_{E}{}^{\boldsymbol{h}}{}_{t} + \boldsymbol{q}^{\boldsymbol{h}}{}_{t}'\mathbf{Z}^{\boldsymbol{h}}{}_{t}|\boldsymbol{I}_t'|\boldsymbol{\lambda}_t]\,. \qquad (AE\text{-}primal_{restricted})$$

$$\begin{aligned} s.t. \qquad \boldsymbol{Y}_t'\,\boldsymbol{\lambda}_t &\geq \boldsymbol{y}^{\boldsymbol{h}}{}_{t} \\ -\boldsymbol{X}_t'\,\boldsymbol{\lambda}_t &\geq -\boldsymbol{x}^{\boldsymbol{h}}{}_{t} \\ \boldsymbol{G}^{\boldsymbol{h}}{}_{t}'\,\boldsymbol{\lambda}_t &\geq \boldsymbol{g}^{\boldsymbol{h}}{}_{t} \\ \boldsymbol{X}_t'\,\boldsymbol{\lambda}_t &= \boldsymbol{x}_{E}{}^{\boldsymbol{h}}{}_{t} \\ \mathbf{1}'\,\boldsymbol{\lambda}_t &\geq 1 \\ \boldsymbol{\lambda}_t &\geq \mathbf{0} \end{aligned}$$

It is remarkable that the objective function has been modified. The problem is now to minimize the total cost for each period, including the investment expenditures.

Its dual model is

$$C^{h}{}_{AE} + \boldsymbol{q}_t^{\boldsymbol{h}}{}'|\boldsymbol{i}_t^{h}| = \max\{\boldsymbol{\beta}^{h}{}_{yt}'\boldsymbol{y}_t^{h} - \boldsymbol{\beta}^{h}{}_{xt}'\boldsymbol{x}_t^{h} + \boldsymbol{\beta}^{h}{}_{kt}'\boldsymbol{g}_t^{h} + \beta^{h}{}_{0t}\}$$

$$s.t. \qquad \boldsymbol{Y}_t\boldsymbol{\beta}^{h}{}_{yt} - \boldsymbol{X}_t\boldsymbol{\beta}^{h}{}_{xt} + \boldsymbol{G}_t\boldsymbol{\beta}^{h}{}_{kt} + \mathbf{1}\beta^{h}{}_{0} \leq \boldsymbol{X}_t\,\boldsymbol{w}^{\boldsymbol{h}}{}_{\boldsymbol{t}} + |\boldsymbol{I}_t|\mathbf{Z}^{\boldsymbol{h}}{}_{\boldsymbol{t}}\boldsymbol{q}^{\boldsymbol{h}}{}_{t} \qquad (\lambda^{h}{}_{t})$$

All the investments take their absolute values.

6.2. Some Further Expositions for the Dynamic Models

Now both of the technical efficiency and the allocative efficiency models take account of cross period constraints. How do these constraints make a firm's intertemporal behaviors different from those in static cases? Let's focus firstly on (19). It can be arranged as

$$\boldsymbol{x}^{\boldsymbol{h}}{}_t\theta^{\boldsymbol{h}}{}_t \text{ -}\boldsymbol{X}_t\text{'}\boldsymbol{\lambda}^h{}_t \geq \mathbf{1}_{\mathrm{N}}\boldsymbol{q}^{\boldsymbol{h}}{}_t\text{'}\mathbf{Z}^{\boldsymbol{h}}{}_t(|\boldsymbol{I}_t\text{'}|\boldsymbol{\lambda}^h{}_t - |\boldsymbol{i}^{\boldsymbol{h}}{}_t|). \tag{21}$$

The right-hand side of the inequality (21) should be positive for each of its elements according to the original constraints of the problem (*TE-primal*). The investment implies an adjustment cost that can be defined in terms of quantities of variable factors becoming *non productive*. For an efficient DMU that does not invest, it must be true that $\boldsymbol{x}^h{}_t = \boldsymbol{X}_t'\boldsymbol{\lambda}^h{}_t$ and $|\boldsymbol{I}_t'|\boldsymbol{\lambda}^h{}_t = \boldsymbol{i}^{\boldsymbol{h}}{}_t = 0$. For an efficient firm that does invest, the quantities of variable inputs should be greater than those that don't in order to compensate the adjustment cost. In this case, we have, from (19), that $\boldsymbol{x}^h{}_t = \boldsymbol{X}_t'\boldsymbol{\lambda}^h{}_t + \mathbf{1}_{\mathrm{N}}\boldsymbol{q}^h{}_t'\,(|\boldsymbol{I}_t'|\boldsymbol{\lambda}^h{}_t - \boldsymbol{i}^h{}_t) > \boldsymbol{X}_t'\boldsymbol{\lambda}^h{}_t$ and the term $\mathbf{1}_{\mathrm{N}}\boldsymbol{q}^h{}_t'\,(|\boldsymbol{I}_t'|\boldsymbol{\lambda}^h{}_t - \boldsymbol{i}^h{}_t)$ corresponds to the adjustment cost. Whenever a firm's performance is not efficient, the term $\boldsymbol{x}^h{}_t$ should be corrected by $\theta^h{}_t\,\boldsymbol{x}^h{}_t$. This explains why (19) plays the role of variable inputs' constraints in the dynamic DEA model for technical efficiency.

Another remarkable difference between our dynamic models and static ones is (20). The original form of this constraint is $[\,|\boldsymbol{I}_t|\,\mathbf{Z}^{\boldsymbol{h}}{}_t\,(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\text{-}\boldsymbol{K}_t]'\boldsymbol{\lambda}^h{}_t \geq [(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\mathbf{Z}^{\boldsymbol{h}}{}_t\,|\boldsymbol{i}_t{}^h| - \boldsymbol{k}_t{}^h]$. To simplify our interpretations, let's also consider the case of positive investments and thus **Z=E**. Take out a single inequality from it in order to highlight its economic sense, for example, the constraint for the *l*th quasi-fixed input,

$$\{[\boldsymbol{I}_t(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\text{-}\boldsymbol{K}_t]'\,\boldsymbol{\lambda}_t\}_{ll} \geq [(r_t+\delta_l)^{-1}i^h{}_{lt}\text{ -}k^h{}_{lt}], \tag{22}$$

This is equivalent to

$$[k^h{}_{lt}\text{ -}(r_t+\delta_l)^{-1}i^h{}_{lt}] \geq \{[\,\boldsymbol{K}_t\text{-}\boldsymbol{I}_t(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}]'\,\boldsymbol{\lambda}^h{}_t\}_{ll}\,.$$

Multiplying both sides with $(1\text{-}\delta_l)$, we can write the left hand side in form of the following series.

$$(1\text{-}\delta_l)k^h{}_{lt}\text{ -}(1\text{-}\delta_l)(r+\delta_l)^{-1}i^h{}_{lt} = (1\text{-}\delta_l)k^h{}_{lt} - \sum_{\tau=1}^{\infty}\,[(1\text{-}\delta_l)/(1+r_t)]^{\tau}\,i^h{}_{lt} \tag{23}$$

$$= k^h{}_{l(t+1)} - i^h{}_{lt} - \sum_{\tau=1}^{\infty}\,[(1\text{-}\delta_l)/(1+r_t)]^{\tau}\,i^h{}_{lt} \tag{24}$$

$$=k^{h}_{l(t+1)}-\sum_{\tau=0}^{\infty}[(1\text{-}\delta_l)/(1+r_t)]^{\tau}\, i^{h}_{lt}.$$

The passageway from (23) to (24) is due to the capital movement law. Then, we may put (22) into the *value* form by multiplying the investment price q^{h}_{lt}.

$$q^{h}_{lt}k^{h}_{l(t+1)}-\sum_{\tau=1}^{\infty}[(1\text{-}\delta_l)/(1+r_t)]^{\tau}(q^{h}_{lt}i^{h}_{lt}) \geq q^{h}_{lt}(1\text{-}\delta_l)\{[\,\boldsymbol{K}_t\text{-}\boldsymbol{I}_t(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}]'\boldsymbol{\lambda}^{h}_{t}\,\}_{ll}. \qquad (25)$$

At the beginning of the initial time $\tau=0$ which corresponding to t, the DMU invests i^{h}_{lt}. The discounted *residual* value of i^{h}_{lt} for the future is equal to the sum of the series. The difference at the left hand side of (25) represents the *productive* part of the lth capital stock during the period t, and therefore a cost of the current production. Being one of the constraints of the cost-minimization problem, (25) means that for DMU^{h}, the productive part of the capital stock is, at least, as great as that of the *best* DMUs, in other words, the DMU^{h}'s production plan is feasible.

Finally, the objective of the allocative efficiency problem is also augmented in the presence of quasi-fixed factors. Now it is to seek the minimum of total cost while the intertemporal adjustment expenditures are considered.

The unique difference between the first two static models and the dynamic ones we have studied in this chapter is that, in the dynamic model, the multipliers $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are *tied* by the intertemporal restrictions.

7. A Comparison with Nemoto and Goto (1999 and 2003) Model

Recall the dynamic DEA model that Nemoto and Goto (1999 and 2003) proposed. Our model bears some resemblances to theirs, but we argue that our model is more general.

First, instead of working on the variable of investment $\boldsymbol{i}_t$, Nemoto and Goto (2003) imposes that, among others that are the same as ours, for each quasi-fixed input, (for simplicity, we assume in this section that there is only one quasi-fixed input):

$$K_t'\boldsymbol{\lambda}_t \leq k_t \tag{26}$$
$$K_{t+1}'\boldsymbol{\lambda}_t \geq k_{t+1}, \tag{27}$$

where K_t is the column vector of the considered quasi-fixed input of the different DMU's and k_t is the considered quasi-fixed input of the considered DMU.

Let δ be the depreciation rate for this quasi-fixed factor. It follows from (26) and (27) that

$$K_{t+1}'\boldsymbol{\lambda}_t\text{-}(1\text{-}\delta)K_t'\boldsymbol{\lambda}_t \geq k_{t+1}\text{-}(1\text{-}\delta)k_t.$$

This means that restrictions (26) and (27) imposed in Nemoto and Goto (2003) imply our constraints when the investments are non-negative, that is,

$$K_t'\boldsymbol{\lambda}_t \leq k_t \tag{28}$$
$$I_t'\boldsymbol{\lambda}_t \geq i_t. \tag{29}$$

where I_t is the column vector of the investment in the considered quasi-fixed input of the different DMU's and i_t is the investment in the quasi-fixed input of the considered DMU.

Note that (28) and (29) do not imply (27). In fact, (28) implies that there exists a non negative γ_t such that $(1\text{-}\delta)K_t'\,\boldsymbol{\lambda}_t+\gamma_t = (1\text{-}\delta)k_t$. This equation and (29) yield

$$K_{t+1}'\,\boldsymbol{\lambda}_t+\gamma_t = [(1\text{-}\delta)K_t + I_t]'\boldsymbol{\lambda}_t + \gamma_t \geq (1\text{-}\delta)k_t+i_t = k_{t+1},$$

but not $K_{t+1}'\lambda_t \geq k_{t+1}$. In other words, our restrictions on the capital movement, (28) and (29), are weaker than those of Nemoto-Goto's and thus the feasible set of Nemoto-Goto's model is a subset of ours. Consequently, the cost function determined in the present paper is smaller than that determined in the way shown in Nemoto and Goto (2003). In general, the production technology determined in this paper is more productive than that drawn from Nemoto and Goto (2003) and hence closer to the theoretical one when the investment is non-negative. Thus, one can conclude that the model of Nemoto and Goto (2003) leads to an overestimated efficiency score in case of non-negative investment.

When investment i_t=[k_{t+1}-(1-δ)k_t] is negative for each DMU, the reasonable constraints on the investment should be $-I_t'\lambda_t \geq -i_t$ since the adjustment cost increases with the disinvestment quantity and the *fictive* DMU suffers from an adjustment cost at least as that of the considered DMU. This inequality is just our restriction. It implies, together with (26), $K_{t+1}'\lambda_t \leq k_{t+1}$, and not (27) (Nemoto-Goto restriction). Consequently, the feasible set of the model of Nemoto and Goto (2003) is incorrect and so for the efficiency score.

A second contribution is worth noting. The model of Nemoto and Goto (2003) includes, for each DMU, one constraint for each input and each output, and that for *every* period. For example, in case of one variable input/one quasi-fixed input/one output, if there are T periods, the model includes 4*T constraints, that is T constraints for the variable input, 2*T constraints for the quasi-fixed inputs, and T constraints for the output (plus one additional constraint if variable returns to scale are imposed). Obviously, this may represent a numerical burden. Furthermore, there is no reason *a priori* to suppose that the time horizon for the firm is equal to the number of period of observations. Also, the interpretation of the results becomes quite troublesome. This model, if we are ready to accept it as such, is meaningful only for the first period. Period 2 to T restrictions are only *expected* (even if we assume perfect foresight). This model has a meaning only if we are ready to assume that what has been planned in period 1 for periods 2 to T is realized as such in reality. In other words, the acquisition of information after period 1 will not change the decision for periods 2 to T taken in period 1. Every decision is taken in period 1. Certainly, this is a very restrictive behavioral assumption. In our model, the decisions for period t are taken in period t, and new information is explicitly introduced.

8. Generalization of Efficiency Measurements

Recall the traditional measures of efficiencies *TE*, *AE* and *OE* previously defined. The measure of technical efficiency of DMU^h in a given year is actually the ratio of the cost of a technically efficient variable input bundle over observed variable cost.

$$TE^h{}_t = \theta^h{}_t.$$

The measure of allocative efficiency of DMU^h is

$$AE^h{}_t = C^h{}_{min\text{-}t} / (TE^h{}_t \times C^h{}_{obs\text{-}t}).$$

And, the measure of global efficiency is defined as the product

$$OE^h{}_t = TE^h{}_t \times AE^h{}_t = C^h{}_{min\text{-}t} / C^h{}_{obs\text{-}t} .$$

The performance of a given DMU is efficient if and only if $OE = 1$.

We have seen that the dynamic and the static allocative models were different: the investment expenditures are taken into the objective function and thus the problem is actually to minimize the total costs of current production, not only the variable costs. This is one of the significant impacts of the intertemporal restrictions. In fact, it is impossible to consider the variable cost independently on and separately from the investment costs. To capture the nature of dynamic context, we should appropriately redefine those efficiency measures.

At this point, the objective function of the dynamic allocative efficiency model AE-*primal* is heuristic. From it, we define now (for DMU *h* at time *t*):

$$AE_d{}^h{}_t = \frac{\text{Optimal total cost}}{\text{Technically efficient total cost}} = \frac{\boldsymbol{w^h{}_t{}'x_E{}^h{}_t + q^h{}_t{}'Z^h{}_t \,|I_t|\,'\lambda^h{}_t}}{TE^h{}_t \times C^h{}_{obs\text{-}t} + \boldsymbol{q^h{}_t{}'Z^h{}_t}\,|\boldsymbol{i}_t{}^h|} .$$

The relevant technical efficiency measure is defined by

$$TE_d{}^h{}_t = \frac{\text{Technically efficient total cost}}{\text{Observed total cost}} = \frac{TE^h{}_t \times C^h{}_{obs\text{-}t} + \boldsymbol{q^h{}_t{}'Z^h{}_t}\,|\boldsymbol{i}_t{}^h|}{C^h{}_{obs\text{-}t} + \boldsymbol{q^h{}_t{}'Z^h{}_t}\,|\boldsymbol{i}_t{}^h|} .$$

The product $TE_d{}^h{}_t \times AE_d{}^h{}_t$ defines the global efficiency measure:

$$OE_d{}^h{}_t = TE_d{}^h{}_t \times AE_d{}^h{}_t$$
$$= \frac{Optimal\ total\ cost}{Observed\ total\ cost}$$
$$= \frac{\boldsymbol{w}^h{}_t{}'\boldsymbol{x}_E{}^h{}_t + \boldsymbol{q}^h{}_t{}'\mathbf{Z}^h{}_t \left|\boldsymbol{I}_t\right|{}'\boldsymbol{\lambda}^h{}_t}{C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\mathbf{Z}^h{}_t \left|\boldsymbol{i}_t^h\right|},$$

where $C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\mathbf{Z}^h{}_t \left|\boldsymbol{i}_t^h\right|$ is the observed total cost.

The three new efficiency measures degenerate to static ones if no investment occurs. Moreover, it is not difficult to see, from the definitions, the following interesting relations between the static measures and dynamic measures.

$$TE_d{}^h{}_t - 1 = (TE^h{}_t - 1) \times [C^h{}_{obs\text{-}t} / (C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\, \boldsymbol{i}_t^h)],$$
$$= (TE^h{}_t - 1) \times (\textit{Share of the observed variable cost in the observed total cost}),$$
$$AE_d{}^h{}_t - 1 = (AE^h{}_t - 1) \times [(TE^h{}_t \times C^h{}_{obs\text{-}t}) / (TE^h{}_t \times C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\, \boldsymbol{i}_t^h)]$$
$$+ \boldsymbol{q}^h{}_t{}'\, (\mathbf{Z}^h{}_t \left|\boldsymbol{I}_t{}'\right| \boldsymbol{\lambda}^h{}_t - \boldsymbol{i}_t^h) / (TE^h{}_t \times C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\, \boldsymbol{i}_t^h),$$
$$= [(AE^h{}_t - 1) \times (\textit{Share of the technically efficient variable cost in the technically efficient total cost})] + (\textit{Share of inefficient investment cost in the technically efficient total cost}),$$
$$OE_d{}^h{}_t - 1 = (OE^h{}_t - 1) \times [C^h{}_{obs\text{-}t} / (C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\boldsymbol{i}_t^h)] + \boldsymbol{q}^h{}_t{}'(\mathbf{Z}^h{}_t \left|\boldsymbol{I}_t\right|{}'\boldsymbol{\lambda}^h{}_t - \boldsymbol{i}_t^h) / (C^h{}_{obs\text{-}t} + \boldsymbol{q}^h{}_t{}'\boldsymbol{i}_t^h),$$
$$= [(OE^h{}_t - 1) \times (\textit{Share of the observed variable cost in the observed total cost})] + (\textit{Share of inefficient investment cost in the observed total cost}).$$

It must be noted that if massive disinvestments occur, the great value of sold or consumed equipment, that enters the cost function in negative values, becomes a part of the DMU's revenue and reduces significantly the value of total cost. As a consequence, the new measures of efficiency $TE_d{}^h{}_t$, $AE_d{}^h{}_t$ and $OE_d{}^h{}_t$ may be greater than one. Moreover, it is possible that these new efficiency measures take negative values. This phenomenon is impossible in static settings. The negative scores and those greater than one contradict somewhat the usual concept: an efficiency score should preferably be between zero and one. Anyway, it is mathematically easy to define an efficiency score in order to conform better to the usual custom. Before to do this, we must answer the following questions.

With dynamic settings, what is the criterion for a given DMU to be efficient, that is, to be on the frontier? Is it still the case that a DMU is efficient if and only if $OE_d = 1$?

An efficient DMU must have a score equal to one, neither less nor greater than one. This is because any score OE_d different from one means some kind of deviation from optimal performance. As the efficiency score may deviate one from the left as well as from the right, our assumption about the symmetry between investment cost and disinvestment cost implies that the scores 1-ε and 1+ε means the same inefficiency level, for any ε > 0. Based on this observation, we introduce the following index of global efficiency:

$$GE^h{}_t = \frac{1}{1+\left|1-OE_d{}^h{}_t\right|}.$$

Obviously, the global efficiency score *GE* is always a positive number between zero and one; a greater *GE* corresponds a more effective DMU; and a DMU is efficient if and only if its *GE* is equal to one. The relation between *GE* and OE_d is illustrated in the following figure.

If there is neither investment nor disinvestment, OE_d does not differ from *OE*. This corresponds to the part of the curve *GE* above the interval [0, 1]. The particular case of *OE* = 0 corresponds to *GE* =1/2, instead of zero. *GE* offers only an alternative choice of the efficiency measurement to conform the custom if this is necessary.

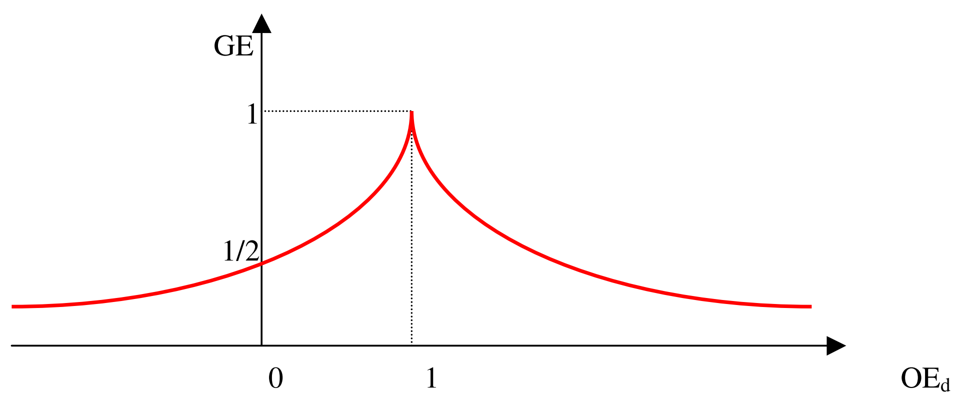

Figure 2 Relationship between GE and OE_d

9. Technology Measurements

9.1. Implicit Prices

The implicit prices of quasi-fixed inputs are just the negative value of $\partial C^h_t/\partial \boldsymbol{k}^h_t$, *i.e.*, $\boldsymbol{\beta}^h_{kt} = -\partial C^h_t/\partial \boldsymbol{k}^h_t{}'$, where $\boldsymbol{\beta}^h_{kt}$ is evaluated at the optimal level and can be obtained from the solution of the dual problem of allocative efficiency.

As for the implicit prices of investments, one needs to get $\boldsymbol{\beta}^h_{kt}$, at first. Then, the implicit prices can be calculated by using the relation

$$\boldsymbol{\beta}^h_{it} = |\,(r_t\mathbf{E}+\boldsymbol{\Delta})^{-1}\boldsymbol{\beta}^h_{kt} - \boldsymbol{q}^h_t\,|,$$

since $\partial C^h_{AE}/\partial\,|\boldsymbol{i}^h|' = \boldsymbol{\beta}^h_{it}$. Precisely, $\beta^h_{it,l} = |\,\beta^h_{kt,l}/(r_t+\delta_l) - q^h_t\,|$, $l = 1, 2, ..., L$.

9.2 Returns to Scale

The solution of the dual problem of technical efficiency gives us $\boldsymbol{\alpha}^h_{yt}$ and $\boldsymbol{\alpha}^h_{xt}$, which are the transposed vectors of the partial derivatives of the production frontier $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$. Of course, one can easily compute returns to scale, shadow prices and potential economies all by using the values of the dual variable of the technical efficiency model. Anyway, one can also choose another alternative method if needed. In fact, by using the relations between $\boldsymbol{\alpha}$'s and $\boldsymbol{\beta}$'s obtained in (9), we can obtain the elasticity of returns to scale with $\boldsymbol{\beta}$'s. Indeed, (6) reflects the duality between the production function and cost function. Substituting the relations (6), we can recover the returns to scale by using any of the following:

$$\begin{aligned} RTS^h &= -\,[F_x(\boldsymbol{x}_E{}^h{}_t) + F_k\boldsymbol{k}^h_t + F_i\boldsymbol{i}^h_t] \div (F_y\boldsymbol{y}^h_t) \\ &= -\{[\boldsymbol{\alpha}^h_{xt}{}'/(\mathbf{1}_N{}'\boldsymbol{\alpha}^h_{xt})]\,\boldsymbol{x}_E{}^h{}_t + [\boldsymbol{\alpha}^h_k{}'/(\mathbf{1}_N{}'\boldsymbol{\alpha}^h_{xt})]\boldsymbol{k}^h_t - [\boldsymbol{\alpha}^h_{it}{}'/(\mathbf{1}_N{}'\boldsymbol{\alpha}^h_{xt})]\boldsymbol{i}^h_t\} \div \{-[\boldsymbol{\alpha}^h_{yt}{}'/(\mathbf{1}_N{}'\boldsymbol{\alpha}^h_{xt})]\,\boldsymbol{y}^h_t\} \\ &= [\boldsymbol{\beta}_E{}'\,\boldsymbol{x}_E{}^h{}_t + \boldsymbol{\beta}_k{}'\,\boldsymbol{k}^h_t - \boldsymbol{\beta}_i{}'\,\boldsymbol{i}^h_t] \div [\boldsymbol{\beta}_y{}'\boldsymbol{y}^h_t] \\ &= [\boldsymbol{w}^h{}'\,\boldsymbol{x}_E{}^h{}_t + \boldsymbol{\beta}_k{}'\,\boldsymbol{k}^h_t - \boldsymbol{\beta}_i{}'\,\boldsymbol{i}^h_t] \div [\boldsymbol{\beta}_y{}'\boldsymbol{y}^h_t]. \end{aligned}$$

10. Conclusions

We have introduced a dynamic version of DEA. The augmented model allows us to measure technical and allocative efficiencies as in the standard model while the inter-temporal restrictions are considered. It has been shown that the necessary data set is made up of contemporary prices and quantities of inputs (including investment) and outputs so that the calculation is fairly easy to perform. Of course, the linear program has exactly the same form than the standard model, so that no particular programming is required. Under dynamic circumstance, we define new measures of efficiency that take account of investment expenditures. Finally, we develop explicit formula of implicit prices of quasi-fixed inputs and investments, and returns to scale.

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